

7.5 The Symmetric and Alternating Groups

Cor 7.22 - Every finite group is isomorphic to a subgroup of S_n

Recall S_n - all bijective functions (permutations)

$$\{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$$

group operation is the composition of functions

Notation for $\sigma \in S_n$: $(\begin{matrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{matrix})$

Cycles - special elements of S_n

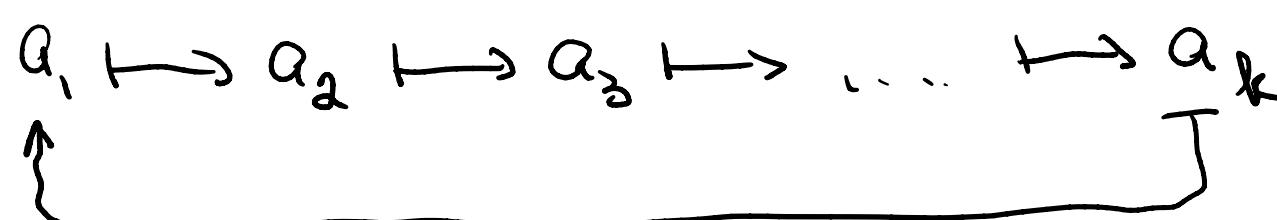
Notation: $\sigma = (a_1 a_2 \dots a_k)$

a_1, \dots, a_k are all distinct

elements of the set $\{1, \dots, n\}$

$$k \leq n$$

Meaning: $\sigma(a_1) = a_2, \sigma(a_2) = a_3, \dots, \sigma(a_{k-1}) = a_k, \sigma(a_k) = a_1$



All other elements stay
in their places:

$$\sigma(b) = b \text{ if } b \neq a_i$$

Easy to see: $|(\alpha_1 \dots \alpha_k)| = k$

- order of this element of S_n

Def $\sigma = (a_1, \dots, a_k)$ and $\tau = (b_1, \dots, b_r)$
are disjoint when they do not share elements.

Ex (125) and (34) are two disjoint cycles in S_5 (also S_{17})

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 4 & 1 \end{pmatrix}$$

Th 7.24 Every permutation can be written as a product
of disjoint cycles in a unique way.

Th 7.23 Disjoint cycles commute:

If $\sigma, \tau \in S_n$ are disjoint cycles, then $\sigma\tau = \tau\sigma$.

Pf $\sigma = (a_1, \dots, a_k)$ $\tau = (b_1, \dots, b_r)$

Let $1 \leq x \leq n$. Wanted: $\sigma(\tau(x)) = \tau(\sigma(x))$

If x is neither a_i nor b_j , then $\sigma(x) = x$; $\tau(x) = x$

$$\sigma(\tau(x)) = \tau(\sigma(x)) = x$$

If $x = a_i$, then $x \neq b_j$. $\sigma(x) = a_l$ ($l = i+1$ or $l = 1$) $\tau(x) = x$

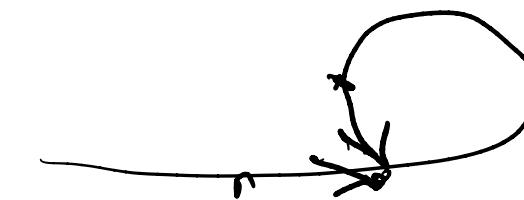
$$\tau(a_i) = a_i$$

$$\begin{aligned} G(\tau(x)) &= G(x) = a_\ell \\ \tau(G(x)) &= \tau(a_\ell) = a_\ell \end{aligned}$$

PF (7.24) $\sigma \in S_n$

$$a_i \in \{1, \dots, n\} \quad a_1 = \sigma(a_1) \quad a_2 = \sigma(a_2) \quad \dots$$

$$a_1 \xrightarrow{\sigma} a_2 \xrightarrow{\sigma} a_3 \dots a_k \xrightarrow{\sigma} a_1$$



would contradict

the injectivity of σ

Pick another element b_1 , not from
this cycle

$$b_1 \xrightarrow{\sigma} b_2 \xrightarrow{\sigma} \dots \xrightarrow{\sigma} b_r \xrightarrow{\sigma} b_1$$

When all elements from $\{1, \dots, n\}$ are either members
of a cycle or are left on their places by σ , we see
that σ is the product of the cycles which we constructed.

Transpositions - cycles of length 2

(a, b)

$a \mapsto b$

$b \mapsto a$

$(\dots a \dots b \dots)$

Th 7.26 Every permutation can be presented as a product of transpositions. (not necessarily disjoint)

Pf It suffices to present a cycle as a product of transpositions:

$$(a_1, \dots, a_k) = (a_1 a_2)(a_2 a_3) \dots (a_{k-1} a_k)$$

$a_1 \mapsto a_2$

$a_1 \mapsto a_3$

$a_1 \mapsto a_2$

$a_2 \mapsto a_3$

$a_k \mapsto a_1$

Def A permutation is EVEN if it can be presented as a product of an even number of transpositions

Otherwise - ODD permutation.

Th 7.28 This parity is invariant (the definition is justified)

$\left\{ \begin{array}{l} \text{Pf} \\ \text{A function in } n \text{ variables is called } \underline{\text{skew-symmetric}} \text{ if} \\ f(\dots x_k, x_{k+1}, \dots) = -f(\dots x_{k+1}, x_k, \dots) \\ 1 \leq k \leq n-1 \end{array} \right.$
All real numbers

Lemma Let f be a skew-symmetric function. Then

$$f(\dots x_i \dots x_j \dots) = -f(\dots x_j \dots x_i \dots)$$

$\left\{ \begin{array}{l} \text{Pf} \\ \text{- Induction in } l = j-i-1 - \text{amount of entries between } x_i \text{ and } x_j \end{array} \right.$

Base $l=0$ - from the definition of skew-symmetric functions

Step Assume the statement is true

for $l < t$, and try to prove it for $l=t$

$$f(\dots \underbrace{x_i x_{i+1} \dots}_{t} x_{j-1} x_j \dots) \quad j-i-1=t$$

$$= -f(\dots \underbrace{x_{i+1} x_i \dots}_{t-1 \text{ - distance}} x_{j-1} x_j \dots)$$

$$= f(\dots x_{i+1} \underset{j}{x_j} \dots x_{j-1} x_i \dots)$$

$$= -f(\dots x_j x_{i+1} \dots x_{j-1} x_i \dots)$$

\equiv

Continuation of the proof of Th 7.28

Let f be a skew-symmetric function

Let σ be an arbitrary permutation

$$f(1, 2, \dots, n) = (-1)^{w(\sigma)} f(\sigma(1), \sigma(2), \dots, \sigma(n))$$

$$(-1)^{w(\sigma)} = \frac{f(\sigma(1), \dots, \sigma(n))}{f(1, 2, \dots, n)}$$

Def $A_n \subset S_n$

alternating group - only even permutations

Th 2.9 A_n is a subgroup of S_n of order $\frac{n!}{2} \rightarrow 6 \times 24$

Let
 $w(\sigma)$ be the
amount of
transpositions
in a presentation
of σ as the
product of
transpositions

A skew-symmetric function in n variables such that $f(1, 2, \dots, n) \neq 0$

Prop 2. Let $\Delta(x_1, \dots, x_n) = \prod_{1 \leq j < i \leq n} (x_i - x_j)$

product over all pairs $(i, j) \mid j < i, i, j \in \{1, 2, \dots, n\}$

1. $\Delta(x_1, \dots, x_n) \neq 0$ if $x_i \neq x_j$ for $i \neq j$ (pairwise distinct)
(In particular, $\Delta(1, 2, \dots, n) \neq 0$)
2. Δ is skew-symmetric